Pricing and Hedging Volatility Derivatives

MARK BROADIE AND ASHISH JAIN

MARK BROADIE
is the Carson Family professor of business at Columbia University in New York, NY.
mnb2@columbia.edu

ASHISH JAIN
is a senior associate at Lehman Brothers in New York, NY.
ashish.jain@lehman.com

This article studies the pricing and hedging of variance swaps and other volatility derivatives, including volatility swaps and variance options, in the Heston stochastic volatility model. Pricing and hedging results are derived using partial differential equation techniques. We formulate an optimization problem to determine the number of options required to best hedge a variance swap. We propose a method to dynamically hedge volatility derivatives using variance swaps and a finite number of European call and put options.

Volatility derivatives are securities whose payoff depends on the realized variance of an underlying asset or index return. Realized variance is the variance of the underlying asset's return over the life of the volatility derivative. A variance swap has a payoff which is a linear function of the realized variance, a volatility swap has a payoff which is a concave function of the realized variance, and a variance call option's payoff is a convex function of the realized variance.

In this article, we propose a methodology for hedging volatility swaps and variance options using variance swaps. Since the prices of both variance swaps and volatility swaps depend on the realized variance of the underlying asset, there must be a relationship between their prices to avoid arbitrage. Because variance swaps can be priced and hedged using actively traded European call and put options, by exploiting the no-arbitrage relationship between volatility derivatives and variance swaps we can price and hedge volatility derivatives.

The volatility of asset prices is an indispensable input in both pricing options and in risk management. Through the introduction of volatility derivatives, volatility is now, in effect, a tradable market instrument. Previously, traders would use a delta-hedged option position as a means to trade volatility, but this does not provide a pure volatility exposure since the return also depends on the underlying stock price. Variance and volatility swaps, however, provide pure exposure to volatility and have become quite popular in the market.

Three different groups of traders in variance and volatility swaps have emerged: directional traders, spread traders, and volatility hedgers. Directional traders speculate on the future level of volatility, while spread traders bet on the difference between realized and implied volatility. In contrast, a volatility hedger typically covers short volatility positions. For example, life insurance companies now offer many products with guaranteed benefits (e.g., variable annuities or with-profits funds) and these expose them to short volatility positions that may be offset by using variance swaps.

Variance and volatility swaps capture the volatility of the underlying asset over a specified time period and are effective hedging instruments for volatility exposure. Based on the demand from volatility traders, the market in volatility and variance swaps has developed

rapidly over the last few years and is expected to grow more in the future. Estimating total trading volume is problematic, as with any OTC market, but recent estimates for daily trading volume on indices are in the region of \$30 million to \$35 million notional. Hence, the pricing and hedging of these derivatives has become an important research problem in academia and industry.

Building on the work of Neuberger [1994], Demeterfi et al. [1999] examined properties of variance and volatility swaps. They showed that variance swaps can be replicated by a static position in European call and put options of all strikes and a dynamic trading strategy in the underlying asset. Brockhaus and Long [2000] provided an analytical approximation for the pricing of volatility swaps. Javaheri, Wilmott, and Haug [2002] discussed the valuation of volatility swaps in the GARCH(1, 1) stochastic volatility model. They used a partial differential equation approach to determine the first two moments of the realized variance and then used a convexity approximation to price the volatility swaps. Lipton [2000] priced volatility swaps using a partial differential equation (PDE) approach. Little and Pant [2001] developed a finite difference method for the valuation of variance swaps in the case of discrete sampling in an extended Black-Scholes framework. Detemple and Osakwe [2000] priced European and American options on the terminal value of volatility when volatility follows a diffusion process. Carr et al. [2005] priced options on realized variance by directly modeling the quadratic variation of the underlying process using a Lévy process. Carr and Lee [2005] priced arbitrary pay-offs of realized variance, provided a zero correlation assumption between stock price process and variance process. Broadie and Jain [2007] show that the convexity correction approximation doesn't provide a good estimate of fair volatility strikes in the Heston stochastic volatility and Merton jump-diffusion models.

In this article, we price variance and volatility swaps when the variance process is a continuous diffusion given by the Heston [1993] stochastic volatility model. We compute fair volatility strikes and price variance options by deriving a partial differential equation that must be satisfied by volatility derivatives. We compute the risk management parameters (greeks) of volatility derivatives by solving a series of partial differential equations. Independently, Sepp [2006] priced options on realized variance in the Heston stochastic volatility model by solving a partial differential equation. We present a numerical method to determine the number of options required to hedge a variance swap. We propose a method to dynamically hedge volatility derivatives using variance swaps and a finite number of European call and put options.

The article is organized as follows. We begin by briefly introducing volatility derivatives in the next section. Then, we present the pricing of volatility swaps and the variance options using a partial differential equation approach in the Heston stochastic volatility model. Next, we present the computation of greeks of volatility derivatives in the Heston stochastic volatility model. Finally, we present an optimization approach to hedge variance swaps using a finite number of options. We also present a dynamic approach to hedge volatility swaps using variance swaps. Concluding remarks are given in the last section.

VOLATILITY DERIVATIVES

Volatility and variance swaps are forward contracts in which one counterparty agrees to pay the other a notional amount, N, times the difference between a fixed level and a realized level of volatility and variance, respectively. The fixed level is called the variance strike for variance swaps and the volatility strike for volatility swaps. Realized variance is determined by the variance of the asset's return over the life of the swap.

The variance swap payoff is defined as

$$(V_d(0, n, T) - K) \times N$$

where $V_d(0, n, T)$ is the realized variance of stock return (defined later) over the life of the contract $[0, T]$, where n is the number of sampling dates, the subscript d is used to emphasize that the variance is computed discretely (i.e., with a finite number of sampling dates, n), K is the variance strike, and N is the notional amount of the swap in dollars. The holder of a variance swap at expiration receives N dollars for every unit by which the stock's realized variance $V_d(0, n, T)$ exceeds the variance strike K. The variance strike is quoted in units of volatility squared (e.g., $(20\%)^2$). For example, suppose an investor takes a long position in a variance swap with strike $(20\%)^2 = 0.04$ and a notional of \$1 million. If, over the life of the contract, the realized variance is $(25\%)^2 = 0.0625$, the investor would make a profit of $(0.0625 - 0.04) * 1{,}000{,}000 = \$22{,}500$.

The volatility swap payoff is defined as

$$(\sqrt{V_d(0,n,T)} - K) \times N$$

where $\sqrt{V_d(0,n,T)}$ is the realized stock volatility (quoted in annual terms and defined later) over the life of the contract, where n is the number of sampling dates, K is the volatility strike, and N is the notional amount of the swap in dollars. The volatility strike, K, is typically quoted in units of percent (e.g., 20%). An investor who is long a volatility swap with strike 20% and a notional of \$1 millon would make a profit of $(0.25 - 0.2) * 1{,}000{,}000 = \$50{,}000$ in the previous example.

The procedure for calculating realized volatility and variance is specified in the derivatives contract and includes details about the source and observation frequency of the price of the underlying asset, the annualization factor to be used in moving to an annualized volatility, and the method of calculating the variance. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the time interval $[0, T]$ into n equal segments of length Δt, i.e., $t_i = iT/n$ for each $i = 0, 1, \ldots, n$. Most traded contracts define realized variance to be

$$V_d(0,n,T) = \frac{AF}{n-1}\sum_{i=0}^{n-1}\left(\ln\left(\frac{S_{i+1}}{S_i}\right)\right)^2 \tag{1}$$

for a swap covering n return observations. Here S_i is the price of the asset at the ith observation time t_i and AF is the annualization factor (e.g., 252 ($=n/T$) if the maturity of the swap, T, is one year with daily sampling). This definition of realized variance differs from the usual sample variance because the sample average is not subtracted from each observation. Since the sample average is approximately zero, the realized variance is close to the sample variance.

We call $V_d(0,n,T)$ the discretely sampled realized variance and $V_c(0,T)$ the continuously sampled realized variance. The floating leg of a variance swap, or discrete realized variance, in the limit approaches the continuously sampled realized variance; that is,

$$V_c(0,T) \equiv \lim_{n\to\infty} V_d(0,n,T) \tag{2}$$

In this article, we price volatility derivatives assuming sampling is done continuously. Broadie and Jain [2007] compute fair variance strikes and fair volatility strikes when realized variance is computed discretely.

A European variance call option gives the holder the right to receive a payoff $V_c(0,T)$ in exchange for paying the strike K at the maturity of the variance call option; that is, its payoff is

$$C_T = \max(V_c(0,T) - K, 0) \times N \tag{3}$$

Similarly, the payoff of the variance put option is

$$P_T = \max(K - V_c(0,T), 0) \times N \tag{4}$$

where N is the notional amount in dollars. Unlike European equity options, the payoff of variance options depends on realized variance $V_c(0,T)$ which is not a traded instrument in the market.

We assume the risk-neutral dynamics of the underlying asset, S_t, follows the Heston [1993] stochastic volatility (SV) model:

$$dS_t = rS_t dt + \sqrt{v_t}S_t(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \tag{5}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma_v\sqrt{v_t}dW_t^1 \tag{6}$$

Equation (5) gives the dynamics of the stock price: S_t denotes the stock price at time t, $\sqrt{v_t}$ is the volatility at time t, and r is the riskless interest rate. Equation (6) specifies the evolution of the variance as a mean-reverting process: θ is the long-run mean variance, κ represents the speed of mean reversion, and σ_v is a parameter which determines the volatility of the variance process. The processes W_t^1 and W_t^2 are two independent standard Brownian motions under the risk-neutral measure Q, and ρ represents the instantaneous correlation between the return and volatility processes. The initial value of the stock price is denoted by S_0 and the variance process by v_0. The variance process v_t, unlike S_t, is unobservable, so it needs to be estimated from data (e.g., option prices or a time series of S_t).

In the SV model, continuous realized variance is given by

$$V_c(0,T) = \frac{1}{T}\int_0^T v_s ds \tag{7}$$

The fair variance strike, K^*_{var}, is defined as the value which makes the contract's net present value equal to zero; that is, it is the solution of

$$E_0^Q[e^{-rT}(V_c(0,T) - K^*_{\text{var}})] = 0 \tag{8}$$

where the superscript Q indicates the risk-neutral measure and the subscript 0 denotes expectation at time $t = 0$. In the SV model, the fair variance strike is given by

$$K^*_{\text{var}} = E\left[V_c(0,T)\right] = E\left(\frac{1}{T}\int_0^T v_s ds\right)$$
$$= \theta + \frac{v_0 - \theta}{\kappa T}(1 - e^{-\kappa T}) \qquad (9)$$

where the last equality follows (e.g., from Broadie and Jain [2007]). The fair volatility strike is defined as the value which makes the contract net present value equal to zero; that is, it solves the equation

$$E_0\left[e^{-rT}(\sqrt{V_c(0,T)} - K^*_{\text{vol}})\right] = 0 \qquad (10)$$

Hence, the fair volatility strike can be expressed as

$$K^*_{\text{vol}} = E\left[\sqrt{\frac{1}{T}\int_0^T v_t dt}\right] = E[\sqrt{V_c(0,T)}] \qquad (11)$$

Using Jensen's inequality, 1 we can obtain an upper bound on the fair volatility strike, as follows:

$$K^*_{\text{vol}} = E_0[\sqrt{V_c(0,T)}] \le \sqrt{E_0\left[V_c(0,T)\right]} = \sqrt{K^*_{\text{var}}} \quad (12)$$

Hence, the fair volatility strike is bounded above by the square root of the fair variance strike. The difference in the square root of the fair variance strike and the fair volatility strike is called the convexity correction. Some authors have obtained an approximation of this convexity correction using Taylor's expansion, but Broadie and Jain [2007] show that it is not necessarily accurate in the SV model. We compute fair volatility strikes by deriving a partial differential equation which exploits a no-arbitrage relationship between variance and volatility swaps. Gatheral [2006] provides a numerical integration approach for computing fair volatility strikes in the SV model.

PRICING VOLATILITY DERIVATIVES

Over the past two decades, the volatility of an underlying stock or an index has developed as an asset class in its own right. Variance swaps are very liquid instruments which can be used to trade volatility and they can be regarded as underlying assets in order to price other volatility sensitive instruments, including volatility swaps, variance options, VIX futures, and so on. Using a no-arbitrage argument, we derive a partial differential equation to price volatility derivatives, compute the fair volatility strike, and price variance call and put options.

Pricing Volatility Swaps

Define X_t^T to be the price process of the floating leg of a variance swap:

$$X_t^T = E_t^Q\left[\frac{1}{T}\int_0^T v_s ds\right]$$

This security price X_t^T depends on the variance, v_s, of the underlying asset from time $t = 0$ until maturity T. It has a payoff at maturity, T, which is the same as the floating leg of a continuous variance swap. At time 0 it represents the fair variance strike,

$$K^*_{\text{var}} = X_0^T \qquad (13)$$

From Equation (9) we know the value of this security at time 0 and we can derive the stochastic differential equation satisfied by the security X_t^T, as follows:

$$dX_t^T = \frac{1 - e^{-k(T-t)}}{kT}\sigma_v v_t dW_t^1 \qquad (14)$$

This price process has zero drift since it is a forward price process. The process X_t^T is driven by the same Brownian motion W_t^1 as the variance process in the SV model. The volatility of the price process, X_t^T, goes to 0 as t approaches T.

Next, we define the price process of a security Y_t^T which represents the floating leg of a volatility swap,

$$Y_t^T = E_t^Q\left[\sqrt{\frac{1}{T}\int_0^T v_s ds}\right]$$

This security has a payoff at time T which depends on the variance process from time $t = 0$ until maturity. At time

T it represents the payoff of the floating leg of the volatility swap. At time 0 it gives the fair volatility strike,

$$K^*_{\text{vol}} = Y_0^T$$

These securities are similar to interest rate derivatives. The price of a zero-coupon bond trading in the market depends on the interest rate process from time 0 until the maturity of the bond. An interest rate is not a tradable market instrument, so for hedging any interest rate product we use other interest rate derivatives which are traded in the market. Similarly, the security Y_t^T depends on the variance process, v_s, which is not a traded instrument in the market. Since the security X_t^T also depends on the variance process, there must be a relationship between the price processes of Y_t^T and X_t^T to avoid arbitrage in the market. Using that relationship we can hedge volatility derivatives using variance swaps.

Next, we define a state variable, I_t, to measure the accumulated variance so far,

$$I_t = \int_0^t v_s ds$$

This state variable is a known quantity at time t and satisfies the differential equation

$$dI_t = v_t dt$$

The forward price process, Y_t^T, can be expressed as

$$Y_t^T = E_t\left[\sqrt{\frac{1}{T}\left(I_t + \int_t^T v_s ds\right)}\right] = F(t, v_t, I_t)$$

and is a function of time, the stochastic variance v_t, and a deterministic quantity I_t. Applying Itô's lemma to $F(\cdot)$ we get

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial v}dv + \frac{\partial F}{\partial I}dI + \frac{1}{2}\frac{\partial^2 F}{\partial v^2}dv^2$$

which can be simplified using Equation (6) to

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial v}\kappa(\theta - v_t) + \frac{\partial F}{\partial I}v_t + \frac{1}{2}\frac{\partial^2 F}{\partial v^2}v_t\sigma_v^2\right]dt + \frac{\partial F}{\partial v}\sigma_v\sqrt{v_t}dW_t^1 \quad (15)$$

Since F is a forward price process, its drift under the risk-neutral measure must be zero. Hence,

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial v}\kappa(\theta - v_t) + \frac{\partial F}{\partial I}v_t + \frac{1}{2}\frac{\partial^2 F}{\partial v^2}v_t\sigma_v^2 = 0 \quad (16)$$

Thus, the forward price process satisfies the partial differential Equation (16) in the SV model. We solve the PDE (16) in the region: $0 \le t \le T$, $I_{\min} \le I \le I_{\max}$, $v_{\min} \le v \le v_{\max}$, with the boundary condition

$$Y_T^T = F(T, v_T, I_T) = \sqrt{\frac{I_T}{T}} \quad (17)$$

At other boundaries (I and V) we set the second order variation of the price process to zero. In particular, we use the following boundary conditions:

$$\left.\frac{\partial^2 F}{\partial I^2}\right|_{(I = I_{\max}, I_{\min})} = 0 \qquad \left.\frac{\partial^2 F}{\partial v^2}\right|_{(v = v_{\max}, v_{\min})} = 0 \quad (18)$$

Thus, by solving Equation (16) with the boundary conditions set forth in Equations (17) and (18), we can compute the fair volatility strike. By solving this partial differential equation, we get the price at all times until maturity. The variance swap forward price process X_t^T satisfies the same differential Equation (16). The boundary condition in the case of a variance swap will be different at maturity and is given by

$$X_T^T = G(T, v_T, I_T) = \frac{I_T}{T} \quad (19)$$

The analytical formula for the variance strike given by Equation (9) solves the PDE (16) with the boundary conditions described in Equations (18) and (19).

Next, we present numerical results to illustrate the computation of fair variance and fair volatility strikes. We use model parameters similar to those estimated in Duffe, Pan, and Singleton [2000] that were found by minimizing mean-squared differences between model and market S&P 500 option prices on November 2, 1993. We adjust the parameters slightly so that the fair continuous variance strike is the same in the Black-Scholes and stochastic volatility models. We assume a risk-free rate of 3.19%. Exhibit 1 gives these parameters. Exhibit 2 shows the fair variance strike and fair volatility strike of a one-year maturity swap computed

EXHIBIT 1

Black-Scholes and SV model Parameters for Pricing and Hedging

Parameters	BS	SV
Correlation ρ	n/a	-0.7
Long-run mean variance θ	n/a	0.019
Speed of mean reversion κ	n/a	6.21
Volatility of variance σ_v	n/a	0.31
Initial volatility $\sqrt{v_0}$	13.261%	10.10%
Risk free rate r	3.19%	3.19%
Real-world growth rate μ	7.0%	7.0%

EXHIBIT 2

Comparison of Fair Variance and Fair Volatility Strikes in the SV Model

The first column shows the fair variance strike computed using the PDE method, simulation, and analytical value in the SV model. The second column shows the fair volatility strikes computed with the same methods.

	$K^*_{\text{var}}(\%^2)$	$K^*_{\text{vol}}(\%)$
Simulation price	13.259	13.094
(standard error)	(0.057)	(0.002)
PDE	13.261	13.096
Analytical	13.261	
Numerical integration		13.096

by solving the PDE (16) with appropriate boundary conditions. We solve the PDE (16) on a three-dimensional grid with 400 points each in the V- and I-directions and 2000 intervals in the t-direction. We also compute the fair variance and fair volatility strikes using Monte Carlo simulation and a numerical integration approach given in Broadie and Jain [2007]. The theoretical value of the fair variance strike is computed using Equation (9). All results are computed under the dynamics of the SV model. We report the square-root of fair variance strike, $\sqrt{K^*_{\text{var}}}$, in the results. The fair variance strike for the parameters in Exhibit 1 is $(13.261\%)^2 = 0.017585$. The results from the PDE approach in this section match the values obtained by other methods.

Exhibit 3 illustrates the dependence of fair variance and fair volatility strikes on initial variance. One advantage of the PDE method over simulation is that we get fair variance and fair volatility strikes for all values of initial variance and accumulated variance. Also, this approach gives prices at all times until maturity. The left graph in Exhibit 3 presents the fair variance strike (plotted as the square root of fair variance strike $\sqrt{K^*_{\text{var}}}$) and the fair volatility strike versus initial volatility $\sqrt{v_0}$. Equation (9) shows that the fair variance strike is a linear function of the initial variance. The fair volatility strike is a not a linear function of the initial variance since its payoff is not a linear function of realized variance. Also, as given by the inequality, presented in Equation (12), the fair volatility strike is less than the fair variance strike.

The *convexity value* is the difference between the square root of the fair variance strike and the fair volatility strike. The right graph in Exhibit 3 plots the convexity value with initial volatility. This illustrates that the convexity value is a decreasing function of initial volatility, $\sqrt{v_0}$.

Pricing Variance Options

The price of a variance call option is given by

$$C_t = E_t^Q\left[e^{-r(T-t)}\max(V_c(0,T)-K,0)\right]\times N \qquad (20)$$

We derive a partial differential equation to price a variance call option using a no-arbitrage argument similar to that in the previous section. The payoff of a variance call can be replicated by continuous trading in a variance swap, and the replicating portfolio gives the price of the variance call option.

We form a portfolio of one variance call option and α units of variance swaps. At time 0, the portfolio value is

$$\Pi_0 = \alpha(X_0^T - K^*_{\text{var}}) + C_0 \qquad (21)$$

This portfolio value is the same as the variance call option value since there is no cost to buy one unit of a variance swap at the inception of the contract. The variance call price process, C_t^T can be represented as

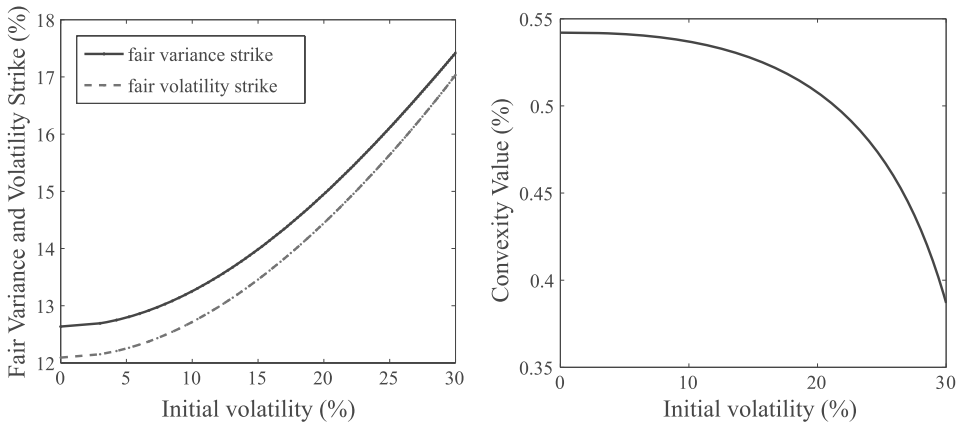
$$C_t^T = G(t, v_t, I_t)$$

Applying Itô's lemma to $G(\cdot)$ we get

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial v}dv + \frac{\partial G}{\partial I}dI + \frac{1}{2}\frac{\partial^2 G}{\partial v^2}\partial v^2 \qquad (22)$$

EXHIBIT 3

The left plot shows the square root of the fair variance strike and the fair volatility strike versus initial volatility. The right plot shows the convexity value (66) versus initial volatility.



which can be simplified using Equation (6) to

$$dG = \left[\frac{\partial G}{\partial t} + \frac{\partial G}{\partial v}\kappa(\theta - v_t) + \frac{\partial G}{\partial I} v_t + \frac{1}{2}\frac{\partial^2 G}{\partial v^2} v_t \sigma_v^2 \right] dt + \frac{\partial G}{\partial v}\sigma_v \sqrt{v_t}\, dW_t^1 \quad (23)$$

From Equation (21), the change in portfolio value in a small time dt is

$$d\Pi_t = \alpha dF + dG \quad (24)$$

Substituting Equations (16), (15), and (23) in Equation (24) and simplifying, we obtain

$$d\Pi_t = \alpha\left(\frac{\partial F}{\partial v}\sigma_v \sqrt{v_t}\, dW_t^1 \right) + \left[\frac{\partial G}{\partial t} + \frac{\partial G}{\partial v}\kappa(\theta - v_t) + \frac{\partial G}{\partial I} v_t + \frac{1}{2}\frac{\partial^2 G}{\partial v^2} v_t \sigma_v^2 \right] dt + \frac{\partial G}{\partial v}\sigma_v \sqrt{v_t}\, dW_t^1 \quad (25)$$

If we choose $\alpha = -\frac{\partial G}{\partial v} \Big/ \frac{\partial F}{\partial v}$, then the stochastic component in the portfolio vanishes and Equation (25) simplifies to

$$d\Pi_t = \left[\frac{\partial G}{\partial t} + \frac{\partial G}{\partial v}\kappa(\theta - v_t) + \frac{\partial G}{\partial I} v_t + \frac{1}{2}\frac{\partial^2 G}{\partial v^2} v_t \sigma_v^2 \right] dt \quad (26)$$

Since the portfolio Π_t is riskless, it should earn the risk free rate of return, and so

$$\left[\frac{\partial G}{\partial t} + \frac{\partial G}{\partial v}\kappa(\theta - v_t) + \frac{\partial G}{\partial I} v_t + \frac{1}{2}\frac{\partial^2 G}{\partial v^2} v_t \sigma_v^2 \right] dt = rG dt \quad (27)$$

which can be rewritten as

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial v}\kappa(\theta - v_t) + \frac{\partial G}{\partial I} v_t + \frac{1}{2}\frac{\partial^2 G}{\partial v^2} v_t \sigma_v^2 - rG = 0 \quad (28)$$

We solve the PDE (28) in the region: $0 \le t \le T$, $I_{\min} \le I \le I_{\max}$, $v_{\min} \le v \le v_{\max}$ with the boundary conditions defined in Equation (18).

We compute the price of variance call and variance put options with a maturity of one year for different strikes. The at-the-money strike is $K = (13.261\%)^2 = 0.017585$ from Exhibit 2. The other strikes are given in Exhibit 4. We use the SV parameters in Exhibit 1. We solve the partial differential equation on a three-dimensional grid with 400 points each in the V- and I-directions and 2000 intervals in the t-direction. We assume a notional $N = \$1000$ in our calculations. Option prices are given in Exhibit 4. When the call and put options are both at the money, their prices are the same due to the put-call parity relationship,

$$C_t - P_t = X_t^T - Ke^{-r(T-t)} \quad (29)$$

EXHIBIT 4

Prices of Variance Call and Put Options in the SV Model

These prices are for one-year maturity options corresponding to the SV model parameters given in Exhibit 1.

Strike K $(\%)^2$	Call ($)	Put ($)
10.272	7.127	0.314
11.095	5.575	0.465
12.581	3.101	1.398
13.261	2.220	2.220
14.527	1.068	4.475
15.691	0.481	7.295

RISK MANAGEMENT PARAMETERS OF VOLATILITY DERIVATIVES

In this section, we compute greeks of variance and volatility swaps using partial differential equations and discuss properties of the greeks. These greeks are required for hedging volatility derivatives. Delta units of volatility derivatives are used to dynamically hedge volatility swaps with variance swaps as explained in the next section. Other greeks are useful in understanding the sensitivity of the price of volatility derivatives to various parameters of the SV model.

Delta of Volatility Derivatives

We define the delta of variance and volatility swaps as the first-order variation in the fair strike with respect to the variance, v_t. Thus, the delta of a variance swap is

$$\dot{v}_t \equiv \frac{\partial X_t^T}{\partial v_t} \tag{30}$$

and the delta of a volatility swap is defined similarly. We compute the delta of a variance swap analytically using Equation (9) to get

$$\dot{v}_t = \frac{\partial X_t^T}{\partial v_t} = \frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{31}$$

The delta of a variance swap is constant and positive since the payoff of the variance swap is a linearly increasing function of realized variance. The delta of the variance swap approaches zero as time to maturity decreases, since at maturity the payoff of the variance swap is independent of the initial variance. We compute the delta of the volatility swap numerically using first-order finite differences. The left plot in Exhibit 5 shows the delta of variance and volatility swaps versus initial volatility.

To make variance and volatility swap deltas comparable, note that

$$\frac{\partial\sqrt{X_0^T}}{\partial v_0} = \frac{1}{2\sqrt{X_0^T}}\frac{\partial X_0^T}{\partial v_0} = \frac{1}{2\sqrt{K_{\text{var}}^*}}\frac{\partial X_0^T}{\partial v_0} \tag{32}$$

and in Exhibit 5, we plot $\partial\sqrt{X_0^T}\big/\partial v_0$ as the variance swap delta and $\partial Y_0^T/\partial v_0$ as the volatility swap delta. We computed volatility swap deltas numerically using first-order finite differences, and we used Equations (31) and (32) to compute $\partial\sqrt{X_0^T}\big/\partial v_0$. This procedure is used for all greeks in the following subsections to make the sensitivities comparable.

Using the parameters from Exhibit 1, the delta of fair variance strike $\partial\sqrt{X_0^T}\big/\partial v_0$, is 60.6%. We approximate the change in the fair variance strike as follows. A change in initial volatility from 10.1% to 11% implies a change in initial variance from 0.010201 to 0.0121, or Δv_0 = 0.001899. The change in the fair variance strike is $\Delta X_0^T = 2 * \sqrt{K_{\text{var}}^*} * \Delta v_0 * 0.606 = 0.000303$. This implies the fair variance strike changes from $(13.261\%)^2 = 0.017586$ to $0.017889 = (13.375\%)^2$. The actual value of fair variance strike at an initial volatility of 11% is $(13.375\%)^2$. As shown in Exhibit 5, the delta of the fair volatility strike is a positive and decreasing function of variance and volatility. Since the volatility swap payoff is a concave function of realized variance, its delta decreases with initial variance and volatility. The right plot in Exhibit 5 shows the difference in the deltas of variance and volatility swaps versus initial volatility.

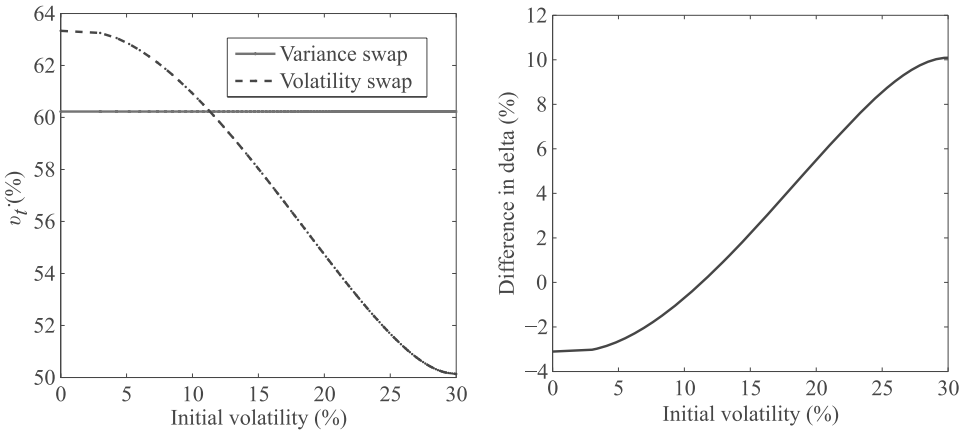
Next, we define the sensitivities of strikes with respect to the parameters of the model.

Volatility Derivatives: $\dot{\kappa}$

We define $\dot{\kappa}$ as the first-order variation in fair strikes with respect to the mean reversion speed, κ. For variance swaps it is defined as

EXHIBIT 5

The left plot shows the sensitivity of fair variance strikes (see Equation (32)) and fair volatility strikes to initial variance as a function of initial volatility. The right plot shows the difference in the deltas of fair variance and volatility strikes.



$$\dot{\kappa} \equiv \frac{\partial X_t^T}{\partial \kappa} \tag{33}$$

Using Equation (9), we get

$$\dot{\kappa} \equiv \frac{\partial X_t^T}{\partial \kappa} = (v_t - \theta)\left(\frac{(T-t)e^{-\kappa(T-t)}}{kT} - \frac{1-e^{-\kappa(T-t)}}{\kappa^2 T}\right) \tag{34}$$

Observe that $\dot{\kappa}$ approaches zero as time to maturity decreases, since at maturity the realized variance is fixed so all the sensitivities must approach zero. We compute the $\dot{\kappa}$ of the volatility swap by differentiating the PDE (16) with respect to the parameter κ

$$\frac{\partial \dot{\kappa}}{\partial t} + \frac{\partial \dot{\kappa}}{\partial v}(\theta - v)\kappa + \frac{\partial F}{\partial v}(\theta - v) + \frac{1}{2}\frac{\partial^2 \dot{\kappa}}{\partial v^2} v\sigma_v^2 = 0 \tag{35}$$

We solve this partial differential equation in the same domain $0 \le t \le T$, $I_{\min} \le I \le I_{\max}$, $V_{\min} \le V \le V_{\max}$ with the boundary conditions,

$$\dot{\kappa}\Big|_{(t=T)} = 0 \tag{36}$$

$$\frac{\partial^2 \dot{\kappa}}{\partial I^2}\Bigg|_{(I=I_{\max}, I_{\min})} = 0 \quad \frac{\partial^2 \dot{\kappa}}{\partial v^2}\Bigg|_{(v=v_{\max}, v_{\min})} = 0 \tag{37}$$

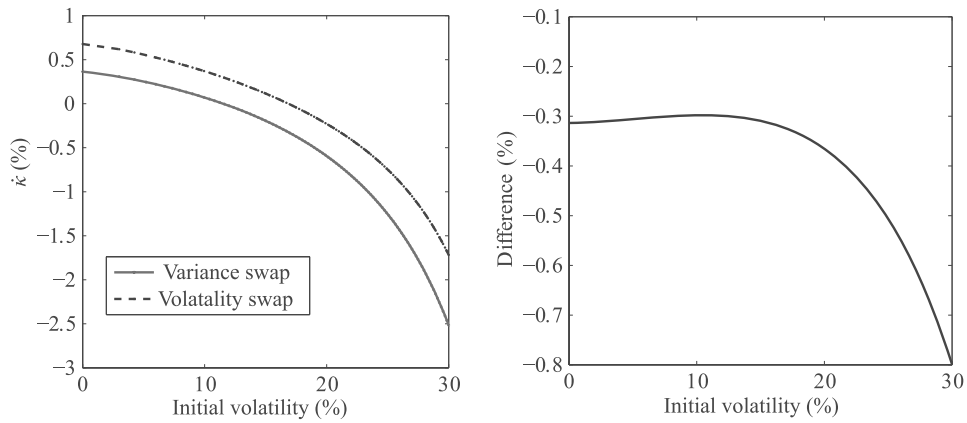
The left plot in Exhibit 6 shows the sensitivity of fair strikes to the mean reversion speed $\dot{\kappa}$ as a function of initial volatility. The variance strike sensitivity plotted in Exhibit 6 is,

$$\frac{\partial \sqrt{X_0^T}}{\partial \kappa} = \frac{1}{2\sqrt{X_0^T}}\frac{\partial X_0^T}{\partial \kappa} = \frac{1}{2\sqrt{K_{\text{var}}^*}}\frac{\partial X_0^T}{\partial \kappa} \tag{38}$$

We plot $\dot{\kappa} = \partial Y_0^T / \partial_\kappa$ for the volatility swap, which we compute by solving the PDE (35). The fair variance strike sensitivity, $\partial\sqrt{X_0^T}\big/\partial\kappa$, to mean reversion speed, κ, is approximately 0.081% = 0.00081 at an initial volatility of $\sqrt{v_0}$ = 10.10%. We compute the approximate change in the fair variance strike if the mean reversion speed, κ, changes from its initial level 6.21 to 7.21 at an initial volatility, 10.1%, as follows. The change in the fair variance strike is $\Delta X_0^T \approx 2 * \sqrt{K_{\text{var}}^*} * \Delta\kappa * 0.00081 \approx 0.000215$. This implies the fair variance strike changes from $(13.261\%)^2 = 0.017586$ to $0.017801 = (13.342\%)^2$. The actual value of the fair variance strike at $\kappa = 7.21$ is $0.017781 = (13.334\%)^2$. The graphs show that the sensitivity changes sign from positive to negative as initial variance increases and the sign change occurs at the long-run mean variance θ. When the initial variance is lower than the long-run mean variance, θ, increasing the mean reversion speed will result in an increase in the variance level, a higher realized variance, and hence a positive $\dot{\kappa}$. The right plot in Exhibit 6 shows the difference in the

EXHIBIT 6

The left plot shows the sensitivity k of fair variance strikes (see Equation (38)) and fair volatility strikes (see Equation (35)) to the mean reversion speed as a function of initial volatility. The right plot shows the difference between the two sensitivities.



sensitivity of the fair variance strike and the fair volatility strike to the mean reversion speed κ versus initial volatility.

Volatility Derivatives: $\dot{\theta}$

We define $\dot{\theta}$ as the first-order variation in the fair strike with respect to the long-run mean variance, θ. For variance swaps it is defined as

$$\dot{\theta} \equiv \frac{\partial X_t^T}{\partial \theta} \tag{39}$$

Using Equation (9), we compute the $\dot{\theta}$ of the variance swap and get

$$\dot{\theta} = \frac{\partial X_t^T}{\partial \theta} = \frac{T-t}{T} - \frac{1-e^{-\kappa(T-t)}}{\kappa T} \tag{40}$$

The fair variance strike sensitivity to the long-run mean variance is constant and positive, since the realized variance increases as the long-run mean variance increases. We compute the $\dot{\theta}$ of the volatility swap by differentiating the PDE (16) with respect to the parameter θ:

$$\frac{\partial \dot{\theta}}{\partial t} + \frac{\partial \dot{\theta}}{\partial v}(\theta - v)\kappa + \frac{\partial F}{\partial v}\kappa + \frac{1}{2}\frac{\partial^2 \dot{\theta}}{\partial v^2} v \sigma_v^2 = 0 \tag{41}$$

We solve this partial differential equation in the same domain $0 \le t \le T$, $I_{\min} \le I \le I_{\max}$, $V_{\min} \le V \le V_{\max}$ with the boundary conditions,

$$\dot{\theta}\Big|_{(t=T)} = 0 \tag{42}$$

$$\frac{\partial^2 \dot{\theta}}{\partial I^2}\Bigg|_{(I = \mathrm{I}_{\max}, \mathrm{I}_{\min})} = 0 \quad \frac{\partial^2 \dot{\theta}}{\partial v^2}\Bigg|_{(v = v_{\max}, v_{\min})} = 0 \tag{43}$$

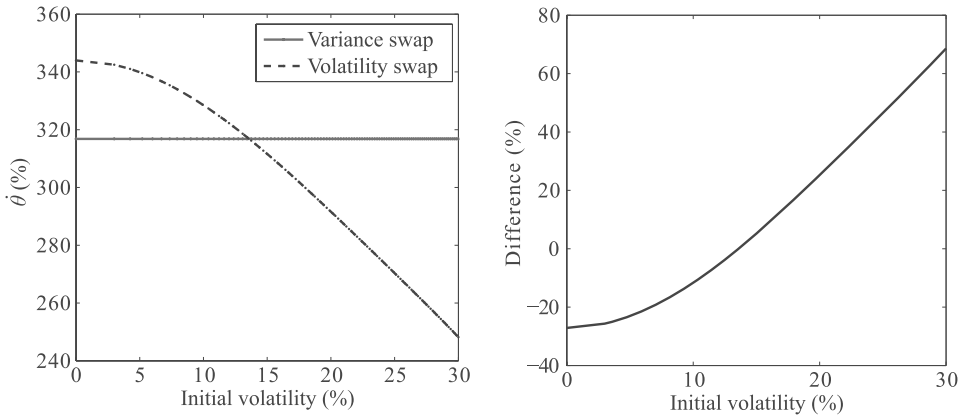
The left plot in Exhibit 7 shows the sensitivity of the fair variance and volatility strikes to the long-run mean variance as a function of initial volatility. As before, we plot the following quantity for variance strike sensitivity:

$$\frac{\partial \sqrt{X_0^T}}{\partial \theta} = \frac{1}{2\sqrt{X_0^T}} \frac{\partial X_0^T}{\partial \theta} = \frac{1}{2\sqrt{K_{\mathrm{var}}^*}} \frac{\partial X_0^T}{\partial \theta} \tag{44}$$

For the fair volatility strike sensitivity we plot $\dot{\theta} = \partial Y_0^T / \partial \theta$, which we compute by solving the PDE (41). Exhibit 7 shows that the fair variance strike sensitivity, $\partial \sqrt{X_0^T} \big/ \partial \theta$, is approximately 316% = 3.16 at an initial volatility of 10.1%. We compute the approximate change in the fair variance strike if the long-run mean variance, θ, changes from 0.019 to 0.021 at an initial volatility 10.1%, as follows. The change in the fair variance strike is $\Delta X_0^T \approx 2 * \sqrt{K_{\mathrm{var}}^*} * \Delta\theta * 3.16 \approx 0.001676$. This implies the fair variance strike changes from $(13.261\%)^2 = 0.017585$ to $0.019265 = (13.879\%)^2$. The actual value of the fair variance strike at $\theta = 0.021$ is $0.019262 = (13.879\%)^2$. The variance swap strike sensitivity to theta is constant at all variance levels. For volatility swaps, $\dot{\theta}$ is positive, which

EXHIBIT 7

The left plot shows the sensitivity $\dot{\theta}$ of fair variance strikes (see Equation (44)) and fair volatility strikes (see Equation (41)) to long-run mean variance $\dot{\theta}$ (39) as a function of initial volatility. The right plot shows the difference between the two sensitivities.



implies that the higher the long-run variance, the higher the fair volatility strike. The $\dot{\theta}$ of the volatility swap is a decreasing function of initial variance. The right plot in Exhibit 7 shows the difference in the sensitivity of the fair variance and volatility strikes to the long-run mean variance as a function of the initial volatility.

Volatility Derivatives: $\dot{\sigma}_\nu$

We define $\dot{\sigma}_\nu$ as the first-order variation in fair strikes with respect to the volatility of variance parameter, σ_ν. For variance swaps it is defined as

$$\dot{\sigma}_\nu \equiv \frac{\partial X_t^T}{\partial \sigma_\nu} \tag{45}$$

Using Equation (9), we find that the fair variance strike is independent of the volatility of variance. We compute the $\dot{\sigma}_\nu$ of the volatility swap by differentiating the PDE (16) with respect to the parameter σ_ν,

$$\frac{\partial \dot{\sigma}_\nu}{\partial t} + \frac{\partial \dot{\sigma}_\nu}{\partial \nu}(\theta - \nu)\kappa + \frac{\partial^2 \dot{\sigma}_\nu}{\partial \nu^2}\nu\sigma_\nu^2 + \frac{\partial^2 F}{\partial \nu^2}\nu\sigma_\nu = 0 \tag{46}$$

We solve this partial differential equation in the domain $0 \le t \le T$, $I_{\min} \le I \le I_{\max}$, $V_{\min} \le V \le V_{\max}$ with the boundary conditions,

$$\dot{\sigma}_\nu\Big|_{(t=T)} = 0 \tag{47}$$

$$\frac{\partial^2 \dot{\sigma}_\nu}{\partial I^2}\Bigg|_{(I=I_{\max}, I_{\min})} = 0 \quad \frac{\partial^2 \dot{\sigma}_\nu}{\partial \nu^2}\Bigg|_{(\nu=\nu_{\max}, \nu_{\min})} = 0 \tag{48}$$

Exhibit 8 shows the sensitivity of fair strikes to the volatility of the variance parameter as a function of initial volatility. Consistent with Equation (9), the fair variance strike is independent of the volatility of variance. The fair volatility strike has a negative dependence on the volatility of variance (i.e., an increase in the volatility of the variance parameter will lead to a decrease in the fair volatility strike). Since the fair volatility strike is a concave function of realized variance, the fair volatility strike decreases with the increase in the volatility of variance parameter, σ_ν. For convex payoff functions (e.g., variance call and put options), the sensitivity with respect to the volatility of variance is positive.

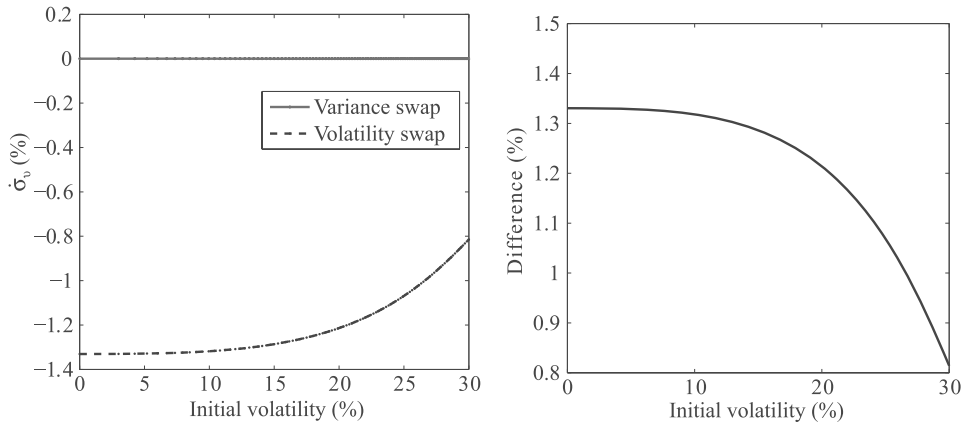
Thus, all the greeks can be computed by either solving the pricing PDE (16) and using finite difference approximations (for delta), or by solving the other related partial differential equations with appropriate boundary conditions.

HEDGING VOLATILITY DERIVATIVES

In this section, we present an approach for hedging volatility derivatives using variance swaps. Other authors (Demeterfi et al. [1999]) have shown that variance swaps

EXHIBIT 8

The left plot shows the sensitivity $\dot{\sigma}_v$ of fair variance strikes and fair volatility strikes to the volatility of variance σ_v (see Equation (45)) as a function of initial volatility. The right plot shows the difference between the two sensitivities.



can be replicated using an infinite number of European call and put options. We formulate an optimization problem to find the best portfolio of European call and put options to closely replicate a variance swap for a given *finite* number of options. We also analyze how replication error decreases as we increase the number of European call and put options in the replicating portfolio. Then, we present an approach to dynamically hedge volatility swaps using variance swaps and a finite number of European call and put options.

Replicating Variance Swaps

In this subsection, we formulate an optimization problem for replicating a variance swap using a static portfolio consisting of a finite number of European call and put options. Applying Itô's lemma to the stock price diffusion (5, 6) we can express realized variance as

$$V_c(0,T) = \frac{1}{T}\int_0^T v_t dt = \frac{2}{T}\left(\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right) \tag{49}$$

This result holds in both the Heston stochastic volatility model and the Black-Scholes model. From Equation (49), the realized variance can be replicated by shorting a log contract and dynamically holding $1/S_t$ shares of stock until the maturity of the contract. Next, we review how to replicate a European-style payoff, in particular, a log contract payoff (Neuberger [1994]), statically using call and put options. Let f be a twice continuously differentiable function which represents the payoff of a European-style path-independent derivative security. It can be expressed as (Breeden and Litzenberger [1978])

$$\begin{aligned} f(S_T) &= f(x) + f'(x)(S_T - x) \\ &\quad + \int_x^\infty f''(K)(S_T - K)^+ dK \\ &\quad + \int_0^x f''(K)(K - S_T)^+ dK \end{aligned} \tag{50}$$

Thus, the payoff function f can be replicated (Carr, Ellis, and Gupta [1998]) by holding positions in a zero-coupon bond with face value $f(x)$, a forward contract with strike x, and call and put options of all strikes using Equation (50). The time-zero value of the claim can be expressed in terms of the European call $C_0(K)$ and put $P_0(K)$ prices of maturity T,

$$\begin{aligned} V_0 &= E_0^Q[e^{-rT} f(S_T)] \\ &= e^{-rT} f(x) + f'(x)[C_0(x) - P_0(x)] \\ &\quad + \int_x^\infty f''(K) C_0(K) dK \\ &\quad + \int_0^x f''(K) P_0(K) dK \end{aligned} \tag{51}$$

Now, let $f(S_T) = \ln(S_T/S_0)$ and $x = S_0$, and substitute in Equation (50) to get

$$\ln\left(\frac{S_T}{S_0}\right) = \frac{S_T - S_0}{S_0} - \int_{S_0}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK - \int_0^{S_0} \frac{1}{K^2}(K - S_T)^+ dK \quad (52)$$

Substituting this in Equation (49) gives

$$V_c(0,T) = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \frac{S_T - S_0}{S_0} + \int_{S_0}^{\infty} \frac{1}{K^2} C_T(K)dK + \int_0^{S_0} \frac{1}{K^2} P_T(K)dK\right] \quad (53)$$

Thus, the floating leg of the variance swap can be replicated (Demeterfi et al. [1999]) by a portfolio having a short position in a forward contract struck at S_0, a long position in $1/K^2$ put options of strike K for all strikes from 0 to S_0, a long position in $1/K^2$ call options for all strikes from S_0 to ∞, and payoffs from a dynamic trading strategy which instantaneously holds $1/S_t$ shares of stock worth \$1 in the portfolio. In particular, Equation (53) shows that continuously realized variance can be replicated in both the Black-Scholes and SV models.

Thus, to replicate the variance swap, we need a short position in a log contract. This can be replicated using call and put options of all strikes as expressed in Equation (52). In practice, we can only form a portfolio using a finite number of options with a limited set of strikes. We analyze how well we can replicate the log contract (and variance swaps) with a finite number of options.

Suppose we want to replicate the log contract with n_p put options and n_c call options of various strikes and common maturity T. We define the portfolio of a log contract and a forward contract as portfolio B. Its payoff at maturity T, when the stock price is S_T, is given by

$$V_B(S_T) = \frac{S_T - S_0}{S_0} - \ln\left(\frac{S_T}{S_0}\right) \quad (54)$$

Let w_i^c represent the number of call options having strike K_i^c and let w_i^p represent the number of put options having strike K_i^p, in portfolio A. The payoff of portfolio A at maturity T when the stock price is S_T is given by

$$V_A(S_T) = \sum_{i=1}^{n_p} w_i^p (K_i^p - S_T)^+ + \sum_{i=1}^{n_c} w_i^c (S_T - K_i^c)^+ \quad (55)$$

where $(K_i^p - S_T)^+$ is the payoff of the put option and $(S_T - K_i^c)^+$ is the payoff of the call option. If we include options of all strikes in portfolio A, then portfolio A exactly replicates portfolio B from Equation (52). The quantities of options in portfolio A are unknown. We compute these values using optimization so that the payoff of portfolios A and B match as closely as possible for a fixed number (n_c and n_p) of call and put options.[2]

To compute the number of options in portfolio A to most closely replicate the payoff portfolio B, we solve the following optimization problem ($P1$)

$$\min_{w^p, w^c} \sum_{j=1}^{n} (V_A(S_T^j) - V_B(S_T^j))^2$$
$$s.t. \quad \sum_{i=1}^{n_p} w_i^p P_0(S_0, K_i^p) + \sum_{i=1}^{n_c} w_i^c C_0(S_0, K_i^c) = P_B(S_0) \quad (56)$$

In the problem ($P1$), the decision variables are vectors w^p, w^c of sizes n_p and n_c, respectively, which represent the quantities of call and put options in the portfolio. The value $V_B(S_T^j)$ is the payoff of the portfolio of log contract and forward contract when the terminal stock price is S_T^j. The value $V_A(S_T^j)$ is the payoff of the portfolio of call and put options when the terminal stock price is S_T^j. The value $P_B(S_0)$ represents the initial value of the portfolio of the log contract and forward contract. The value $P_0(S_0, K_i^p)$ represents the initial value of the put option with strike K^p_i, and $C_0(S_0, K_i^c)$ represents the initial value of the call option with strike K_i^c. The objective function in ($P1$) minimizes the sum of squared differences in two portfolio payoffs at the maturity T over n scenarios. The constraint enforces the initial values of both portfolios to be equal. Thus, the portfolio optimization problem ($P1$) attempts to make the payoffs of the two portfolios as close as possible given the constraint that the initial value of the two portfolios must be equal.

To illustrate, we use the Black-Scholes and SV parameters in Exhibit 1, and set the maturity to be one year. We choose the strikes of the call and put options to be

equally distributed in a three-standard-deviation range defined as follows. For $S_0 = \$100$, we choose the put strikes to be equally distributed between $S_0 e^{(rT - 1/2\sigma^2 T + 3\sigma\sqrt{T})} = \$68$ and \$100. Here, we have chosen $\sigma = \sqrt{K^*_{var}} = 13.26\%$. Thus, for n_p put options in the ($P1$), the put strikes are $K_i^p = 68 + (i-1)(100-68)/(n_p - 1)$, $i = 1, \ldots, n_p$. Similarly, we choose call strikes to be equally distributed between \$100 and $S_0 e^{(rT - 1/2\sigma^2 T + 3\sigma\sqrt{T})} = \$152$. The call strikes are $K_i^c = 100 + (i-1)(152-100)/(n_c - 1)$, $i = 1, \ldots, n_c$. We choose the n scenarios within a four-standard-deviation range. In particular, we take $n = 200$ scenarios of stock prices, $S_T^j = 60 + (j-1)(173-60)/(n-1)$, $j = 1, \ldots, n$. The strike range is narrower than the range used in the optimization scenarios to reflect the practical reality that not all strikes are actively traded in the market.

To compare the performance of the replicating portfolio of call and put options, we compute three types of error,

$$e_1: \quad \frac{E\left|V_A(S_T) - V_B(S_T)\right|}{P_B(S_0)} \tag{57}$$

$$e_2: \quad \frac{\sqrt{E(V_A(S_T) - V_B(S_T))^2}}{P_B(S_0)} \tag{58}$$

$$e_\infty: \quad \frac{\max\left|V_A(S_T) - V_B(S_T)\right|}{P_B(S_0)} \tag{59}$$

where $P_B(S_0)$ represents the value of portfolio B, given in Equation (54), at $t = 0$ when the stock price is S_0. The expectation is under the real-world probability measure.

The objective function in ($P1$) uses equally weighted scenarios, while the error measures, e_1 and e_2, weight the scenarios by their real-world probabilities, so that extreme outcomes have less effect on the results. The error measure e_∞ will be determined by the single scenario with the most extreme outcome. We use equal weighting for scenarios in the objective function so that the portfolio will perform reasonably well under all three error measures. By solving one optimization problem instead of three, we reduce the number of results presented. If the user is interested in one particular error measure, that measure should be substituted in ($P1$).

We solve ($P1$) by forming the Lagrangian and solving the resulting system of linear equations. The solution of optimization problem ($P1$) gives quantities of call and put options to replicate a log contract for a given number of puts and calls in the replicating portfolio. Using this portfolio of call and put options, we analyze the dynamic replication of a variance swap using a finite number of options. We compute these error measures by simulating scenarios in the Black-Scholes and SV real-world probability measures. The value of the drift in the real-world probability measures from Exhibit 1 is $\mu = 7\%$.

Equation (53) gives the formula to replicate continuous realized variance using a portfolio of call and put options and continuous rebalancing of $1/S_t$ shares of stock. There are two types of errors in replicating a variance swap with a finite number of options. The first type of error comes from replicating a log contract by a finite number of options. The second type of error comes from the discrete rebalancing of $1/S_t$ shares of stock worth \$1 in the portfolio. In practice, realized variance is computed for a discrete sampling frequency (e.g., daily or weekly). In order to isolate the effects of the sampling frequency, we present results for the replication of a discrete variance swap for two sampling frequencies: a high sampling frequency of 16 times per day and a daily sampling frequency. In the first case, there are $n = 4096$ sampling observations during the life of a $T = 1$-year variance swap (i.e., 16 samples per day assuming 256 trading days in a year). The same time interval ($\Delta t = 1/4096$ years) is used to compute the payoffs from the dynamic trading strategy. In this case, the majority of the replication error is due to a finite number of options. These results are given in Exhibits 9 and 10.

In Exhibits 9 and 10, a static portfolio of options determined from the solution of ($P1$) is used to replicate the log contract. When computing the variance swap replication errors, the portfolio A payoff is the discrete realized variance payoff at maturity, and the portfolio B payoff is composed of the payoffs from the options portfolio, a short forward contract, and dynamic rebalancing with the stock as given by Equation (53). To normalize the results, we set $P_B(S_0)$ to K^*_{var} (Equation (9)) in the three error measures. To compute the error measures, we simulated 10,000 stock price paths under the real-world measures in the Black-Scholes and SV models. The results show all three error measures decrease as we increase the number of options in the replicating portfolio.[3] With 16 options (8 puts and 8 calls in the option portfolio), the mean absolute replication error for the Black-Scholes model is about 1.9% of the initial value of the portfolio and 2.0% for the SV model. Exhibit 10 shows replication errors and number of options on log scale with a different number of options in the Black-Scholes and SV

EXHIBIT 9

Error in Dynamic Replication of Variance Swap with Finite Number of Options

The first column shows the number of options used in replicating a continuous (almost) variance swap of a one-year maturity. The second column shows the error measure defined in Equation (57) (divided by the variance strike to normalize the error) in the Black-Scholes model and the third column shows the error measure defined in Equation (58). The fourth column shows the error measure defined in Equation (59). The fifth, sixth, and seventh columns show the respective error measures in the SV model. These error measures are computed for a time interval of one year for a variance swap in which realized variance is computed using (n = 4096) sampling observations and rebalancing is done every dt = 1/4096 year in computing payoffs from a dynamic trading strategy.

Number of Options	Black-Scholes			SV		
	Error e_1	Error e_2	Error e_∞	Error e_1	Error e_2	Error e_∞
4	0.381	0.483	1.574	0.392	0.494	5.024
8	0.091	0.115	0.280	0.093	0.128	4.248
16	0.019	0.024	0.120	0.020	0.050	3.411
32	0.004	0.006	0.077	0.005	0.037	2.986

models. These results show error measures, e_1 and e_2, converge quadratically to zero as the number of options increases.

Next, we analyze the effect of a daily sampling interval. We assume that the sampling interval in computing the realized variance is the same as the rebalancing interval in the dynamic trading strategy. Exhibit 11 shows the daily sampling results using a finite number of options. The error measures in Exhibits 9 and 11 are very similar, indicating that daily rebalancing is as effective as more frequent rebalancing, since the error is dominated by using a finite number of options.

Hedging Volatility Derivatives in the SV Model

In this subsection, we present an approach to dynamically hedge volatility swaps using variance swaps in an SV model. Suppose we take a long position in one unit of volatility swap at $t = 0$ of maturity T with fair volatility strike, K^*_{vol}. The volatility swap is initially costless. At time t, the value of the volatility swap contract is

$$\begin{aligned} P_t &= E_t(e^{-r(T-t)}(\sqrt{V_c(0,T)} - K^*_{vol})) \\ &= e^{-r(T-t)}(Y_t^T - K^*_{vol}) \end{aligned} \tag{60}$$

EXHIBIT 10

Replication Error in the B–S and SV Models

Replication error in a variance swap as a function of the number of options. The figures show error measures defined in Equations (57) and (58) in replicating a variance swap with a finite number of options in the Black-Scholes (B-S) and SV models. These error measures are computed for a time interval of one year for a variance swap in which realized variance is computed using (n = 4096) sampling observations and rebalancing is done every dt = 1/4096 year in computing payoffs from a dynamic trading strategy. The figures are plotted on a log-log scale.

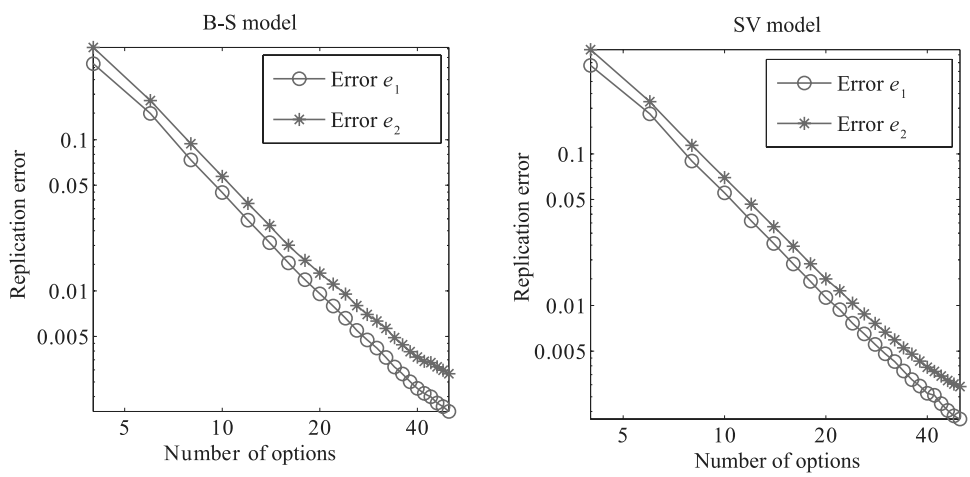


EXHIBIT 11

Error in the Dynamic Replication of Discrete Variance Swap with Finite Number of Options

The first column shows the number of options used in replicating a discrete variance swap of maturity one-year with daily sampling. The rebalancing interval in computing payoffs from the dynamic trading in the underlying stock is daily as well. The second column shows the error measure defined in Equation (57) (divided by the variance strike to normalize the error measures) in the Black-Scholes model and the third column shows the error measure defined in Equation (58). The fourth column shows the error measure defined in Equation (59). The fifth, sixth, and seventh columns show the respective error measures in the SV model.

Number of Options	Black-Scholes			SV		
	Error e_1	Error e_2	Error e_∞	Error e_1	Error e_2	Error e_∞
4	0.381	0.485	1.573	0.393	0.496	5.032
8	0.090	0.114	0.280	0.096	0.127	4.254
16	0.019	0.024	0.124	0.019	0.050	3.421
32	0.004	0.005	0.046	0.005	0.036	2.986

We assume the notional amount of the swap to be \$1. To hedge a long position in volatility swap at time t, we construct a portfolio having one unit of volatility swap and β_t units of variance swaps. Thus the portfolio value at time t equals

$$\begin{aligned}\Pi_t &= E_t\Big[e^{-r(T-t)}(\beta_t(V_c(0,T)-K^*_{\text{var}})) \\ &\qquad + (\sqrt{V_c(0,T)}-K^*_{vol})\Big] \\ &= e^{-r(T-t)}(\beta_t(X_t^T-K^*_{\text{var}})+(Y_t^T-K^*_{vol})) \qquad (61)\end{aligned}$$

The change in this portfolio in a small amount of time dt is given by

$$d\Pi_t = r\Pi_t dt + e^{-r(T-t)}(\beta_t dX_t^T + dY_t^T)$$

which can be written using equation (15) as

$$\begin{aligned}d\Pi_t &= r\Pi_t dt + e^{-r(T-t)}\Bigg[\beta_t\Bigg(\Bigg[\frac{\partial X_t^T}{\partial t}+\frac{\partial X_t^T}{\partial v}\kappa(\theta-v_t) \\ &\quad + \frac{\partial X_t^T}{\partial I}v_t + \frac{1}{2}\frac{\partial^2 X_t^T}{\partial v^2}v_t^2\sigma_v^2\Bigg]dt \\ &\quad + \frac{\partial X_t^T}{\partial v}\dot{\sigma}_v\sqrt{v_t}dW_t^1\Bigg)+\Bigg[\frac{\partial Y_t^T}{\partial t}+\frac{\partial Y_t^T}{\partial v}\kappa(\theta-v_t) \\ &\quad + \frac{\partial Y_t^T}{\partial I}v_t + \frac{1}{2}\frac{\partial^2 Y_t^T}{\partial v^2}v_t^2\sigma_v^2\Bigg]dt + \frac{\partial Y_t^T}{\partial v}\sigma_v\sqrt{v_t}dW_t^1\Bigg]\end{aligned}$$

Since the processes X_t^T and Y_t^T satisfy the pricing PDE (16), the dt terms in the previous equation vanish. Hence, the change in the portfolio value can be rewritten as

$$\begin{aligned}d\Pi_t &= r\Pi_t dt + e^{-r(T-t)}\Bigg[\beta_t\frac{\partial X_t^T}{\partial v}\sigma_v\sqrt{v_t}dW_t^1 \\ &\quad + \frac{\partial Y_t^T}{\partial v}\sigma_v\sqrt{v_t}dW_t^1\Bigg] \qquad (62)\end{aligned}$$

We define β_t as the volatility swap hedge ratio

$$\beta_t = -\frac{\partial Y_t^T/\partial v}{\partial X_t^T/\partial v} \qquad (63)$$

If we choose β_t as in Equation (63), the stochastic component of the portfolio vanishes and the portfolio value is hedged. Thus, for hedging a volatility swap we can take a short position in β_t units of the variance swap, and the portfolio value is hedged dynamically.

Next, we present numerical results for the volatility swap hedging performance. We compute the profit and loss of two different hedging strategies and compare to no hedging. The two hedge portfolios are a portfolio containing one unit of volatility swap and β_t (from Equation (63)) units of a variance swap, and a portfolio containing one unit of volatility swap and a portfolio of European call and put options which replicates β_t units of variance swaps. We compute the portfolio of call and put options which replicates a variance swap as described in the previous subsection.

No hedging: We price the variance and volatility swap of a one-year maturity using the partial differential equation described in the Subsection *Pricing Volatility Swaps*. We also compute the deltas at time zero of the variance and volatility swaps using Equation (30). Together these give the hedge ratio at time zero. We generate 4,800 scenarios of the stock price and variance level at $t = 1/252$ years. The variance and volatility swaps are initially costless. The profit and loss of a long position in an unhedged volatility swap at $t = 1/252$ years is equal to the price of the following volatility swap contract:

$$\begin{aligned}P_t &= E_t(e^{-r(T-t)}(\sqrt{V_c(0,T)}-K^*_{vol})) \\ &= e^{-r(T-t)}(Y_t^T-K^*_{vol}) \qquad (64)\end{aligned}$$

Hedging with variance swaps: We form a portfolio containing one unit of a volatility swap and β_0 units of variance swaps. The value of this portfolio is zero at $t = 0$,

$$\Pi_0 = \beta_0 (X_0^T - K_{\text{var}}^*) + (Y_0^T - K_{vol}^*)$$

where $\beta_0 = -\frac{\partial Y_0^T}{\partial v} \Big/ \frac{\partial X_0^T}{\partial v}$ the hedge ratio. The profit and loss of this hedged portfolio at $t = 1/252$ years is equal to the value of this portfolio,

$$\begin{aligned}\Pi_t &= E_t\Big[e^{-r(T-t)}(\beta_0(V_c(0,T) - K_{\text{var}}^*)) \\ &\qquad + (\sqrt{V_c(0,T)} - K_{vol}^*)\Big] \\ &= e^{-r(T-t)}(\beta_0(X_t^T - K_{\text{var}}^*) + (Y_t^T - K_{vol}^*)) \qquad (65)\end{aligned}$$

Hedging with options: We form a portfolio containing one unit of volatility swap and β_0 units of a portfolio of call and put options which replicates a variance swap. We replicate a variance swap using a portfolio of call and put options, as described in the previous subsection. In these results we are replicating a continuous variance swap with a portfolio of put and call options and a forward contract and payoff from a dynamic trading strategy which holds $1/S_t$ shares of stock (see Equation (53)). For simplicity, we have assumed that rebalancing is done continuously in computing payoffs from the dynamic trading strategy. In this hedging exercise, we present results using 8 (4 calls and 4 puts) options and 32 (16 calls and 16 puts) options. We compute the profit and loss of this portfolio at time $t = 1/252$ years in both cases: hedging with 8 options and hedging with 32 options.

Exhibit 12 shows the performance of hedging volatility swaps with variance swaps and a finite number of options. We compute the error measures e_1, e_2 and e_∞ (Equations (57), (58), and (59), respectively) of profit and loss using 4,800 scenarios of stock prices and variance levels at $t = 1/252$ years. In these results the error measures are normalized by the variance strike, K_{var}^*, defined in Equation (9). We use a batching method with twelve batches to compute the standard error estimates in error measures. The hedging errors are for hedging over an interval of 1/252 years compared to Exhibit 9 where the hedging interval is one year.

From Exhibit 12, we can see that the absolute value of the volatility swap profit and loss is about 5.29% of the variance strike, K_{var}^*, over a single day. Hedging a volatility swap with a variance swap reduces this to 0.03%, which is quite significant. Hence, a volatility swap can be effectively hedged using variance swaps in a dynamic manner. The results also show that hedging with 8 options reduces the absolute value of the profit and loss to 0.15%, and with 32 options to 0.059%. The error in hedging volatility swaps with options decreases as we increase the number of options. Dynamic hedging of volatility swaps with variance swaps and options is shown to be quite effective.

EXHIBIT 12

Hedging a Volatility Swap Using Variance Swaps and Finite Number of Options

The second, third, and fourth columns show the error measures e_1, e_2, and e_∞ (Equations (57), (58), and (59), respectively) of profit and loss in different hedging strategies for a hedging volatility swap over an interval of 1/252 years. The first row shows the mean and standard error of profit and loss of an unhedged volatility swap. The second row shows the mean and standard error of profit and loss of a hedged volatility swap using variance swaps. The third row shows the mean and standard error of profit and loss of a hedged volatility swap with 8 options. The fourth row shows the mean and standard error of profit and loss of a hedged volatility swap with 32 options.

	Error e_1 Avg. (SE) (%)	Error e_2 Avg. (SE) (%)	Error e_∞ Avg. (SE) (%)
Volswap Unhedge	5.293(0.066)	6.646(1.044)	21.811(0.745)
Varswap Hedge	0.029(0.001)	0.042(0.001)	0.289(0.021)
Options Hedge (8)	0.146(0.002)	0.184(0.031)	0.617(0.018)
Options Hedge (32)	0.057(0.001)	0.076(0.014)	0.342(0.018)

CONCLUSION

In this article, we presented a partial differential equation approach to price volatility derivatives in Heston's SV model. The pricing of volatility derivatives (including volatility swaps and variance options) is complicated because the underlying variable, realized variance, is not a market-traded instrument. We exploited a no-arbitrage relationship between variance swaps and other volatility derivatives to derive a partial differential equation to price volatility derivatives. We also derived PDEs for the greeks in order to hedge volatility derivatives. We presented an optimization model for the practical hedging of variance swaps using a finite number of options. We also presented an approach to hedge volatility derivatives using variance swaps, and showed the hedge to be very effective.

ENDNOTES

The article underlying this article was presented at the Fall 2006 INFORMS annual conference, the London Business School, the 2005 Winter Simulation Conference, and at the Columbia Business School. This work was partly supported by NSF grant DMS-0410234.

[1]The concave square-root function Jensen's inequality is

$$E(\sqrt{x}) \leq \sqrt{E(x)}$$

[2]We have chosen portfolio A to hold both call and put options. We can also choose this portfolio to consist of call options or put options, since we can replace the put options by call options and stock using put-call parity.

[3]The error e_∞ does not decrease as fast, because it is sensitive to a single simulation scenario which may lie outside the range of scenarios used in the optimization.

REFERENCES

Breeden, D., and R. Litzenberger. "Prices of State Contingent Claims Implicit in Option Prices." *Journal of Business*, Vol. 51, No. 6 (1978), pp. 621–651.

Broadie, M., and A. Jain. "Discrete and Continuously Sampled Volatility and Variance Swaps." Working paper, Columbia University, 2007.

Brockhaus, O., and D. Long. "Volatility Swaps Made Simple." *Risk*, Vol. 19, No. 1 (2000), pp. 92–95.

Carr, P., K. Ellis, and V. Gupta. "Static Hedging of Exotic Options." *Journal of Finance*, Vol. 53, No. 3 (1998), pp. 1165–1190.

Carr, P., H. Geman, D. Madan, and M. Yor. "Pricing Options on Realized Variance." *Finance and Stochastics*, Vol. 9, No. 4 (2005), pp. 453–475.

Carr, P., and R. Lee. "Robust Replication of Volatility Derivatives." Working paper, Courant Institute and University of Chicago, 2005, http://math.nyu.edu/research/carrp/research.html.

Demeterfi, K., E. Derman, M. Kamal, and J. Zou. "A Guide to Volatility and Variance Swaps." *Journal of Derivatives*, 4 (1999), pp. 9–32.

Detemple, J.B., and C. Osakwe. "The Valuation of Volatility Options." *European Finance Review*, Vol. 4, No. 1, pp. 21–50.

Duffie, D., J. Pan, and K. Singleton. "Transform Analysis and Asset Pricing for Affine Jump Diffusions." *Econometrics*, 68 (2000), pp. 1343–1376.

Gatheral, J. *The Volatility Surface: A Practitioner's Guide*. New York, NY: John Wiley and Sons, Inc., 2006.

Heston, S. "A Closed-Form Solution of Options with Stochastic Volatility with Applications to Bond and Currency Options." *The Review of Financial Studies*, Vol. 6, No. 2 (1993), pp. 327–343.

Javaheri, A., P. Wilmott, and E. Haug. "Garch and Volatility Swaps." Working paper, 2002, http://www.wilmott.com.

Lipton, A. "Mathematical Methods for Foreign Exchange: A Financial Engineer's Approach." *World Scientific*, 2000.

Little, T., and V. Pant. "A Finite Difference Method for the Valuation of Variance Swaps." *Journal of Computational Finance*, Vol. 5, No. 1 (2001), pp. 81–103.

Neuberger, A.J. "The Log Contract." *Journal of Portfolio Management*, Vol. 20, No. 2 (1994), pp. 74–80.

Sepp, A. "Pricing Options on Realized Volatility in Heston Model with Volatility Jumps." Working paper, Bears Stearns, Inc., and Institute of Mathematical Statistics at University of Tartu, 2006.